# Decoherence and Universality of a Quantum Phase Gate

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We study the dephasing noisy effects on a Quantum Phase Gate and the consequences on its universality. To this end we employ two frameworks displaying their differences and analogies.

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A quantum computer processes quantum information that is stored in quantum bits (qubits) [1]. If a small set of fundamental operations, or *universal quantum logic gates* can be performed on the qubits, then a quantum computer can be programmed to solve an arbitrary problem [2]. In particular it was shown [2] that a "controlled-controlled rotation" gate was *universal*. This gate has three qubit inputs and three qubit outputs: the first two inputs go through unchanged, while the third bit is rotated by an angle that is irrationally related to $\pi$ if and only if the first two inputs are 1. Repeated application of this gate allows one to come as close as one wants to a "controlled-controlled NOT" gate that flips the third input if and only if the first two inputs are 1. Controlled-controlled NOT is a *classical universal gate* capable of performing any logic operations. More recently, it was shown [3] that almost any quantum logic gate, with two or more inputs, is universal [3]. In particular, a "controlled rotation" constitutes a single universal quantum gate. Thus, the Quantum Phase Gate (QPG) of any dimension plays an important role in quantum information processing.

However, universality refers to *reversible computation*, i. e. achieved by unitary transformations [4]. Once decoherence affects elementary operations the entire computation is compromised [5]. Here, we address the problem of decoherence and universality in a QPG. In particular we analyze two approaches [6, 7] describing dephasing noisy effects on a QPG.

A gate operation ideally corresponds to a physical process $|\Psi_{\rm out}\rangle = U|\Psi_{\rm in}\rangle$ where a given input state is mapped to an output state by a unitary transformation $U$.

Let us consider the Hilbert space $\mathcal{H}$ of $N$ qubit ($N \geq 2$), where $\mathcal{B} \equiv \{|1\rangle, |2\rangle, \dots |2^N\rangle\}$ are orthogonal states spanning the entire $\mathcal{H}$ space. Then, a QPG can be represented in $\mathcal{B}$ by the $2^N \times 2^N$ matrix

$$U = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & 0 & e^{-it} \end{pmatrix}. \qquad (1)$$

We simply denote the phase by $t$ since it is always related to a physical process, hence it could be replaced by properly scaled time.

For $N = 2$ the transformation (1) can be realized, for instance, by considering a crossed Kerr-like Hamiltonian, $H = \chi a^\dagger a b^\dagger b$ coupling two modes, $a$ and $b$, having different polarization [8]. Instead, for $N = 4$, the transformation (1) easily allows to construct maximum entangled states [9].

Now, provided $t \neq 0$, we know that the logic operation (1) is universal [3]. Then, it would be instructive to consider e. g. the realization of a controlled not (CNOT) gate by repeated application of QPG. The CNOT gate can be represented in $\mathcal{B}$ by

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$$U = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & 0 & -1 \end{pmatrix}, \quad (2)$$

together with single qubit rotations. Of course, once $t = \pi$, (1) and (2) coincide. However, we want to consider the case of realizing the desired transformation (2) in several steps, i. e. by composing several transformations (1), especially because the achievable conditioned phase shift is usually quite small [10].

To characterize a gate process we proceed as follows. We assume the input state to be

$$|\Psi_{\text{in}}\rangle\langle\Psi_{\text{in}}| \Longleftrightarrow \rho_{\text{in}} = \sum_{i,j=1}^{2^N} c_i c_j^* |i\rangle\langle j| \,. \quad (3)$$

Then, the output state can be written as

$$|\Psi_{\text{out}}\rangle\langle\Psi_{\text{out}}| \Longleftrightarrow \rho_{\text{out}} = \sum_{i,j=1}^{2^N} c_i c_j^* R_{j,i} \,, \quad (4)$$

where $R_{j,i} = U|i\rangle\langle j|U^\dagger$ .

Whenever the input-output transformation $U$ is not unitary, we can account for the dephasing noisy effects by averaging over a suitable probability distribution $P$, that is

$$R_{j,i} \longrightarrow \overline{R}_{j,i} = \int \mathrm{d}t' P(t,t') U(t')|i\rangle\langle j|U^\dagger(t') \quad (5)$$

or, equivalently

$$\rho_{\text{out}} \longrightarrow \overline{\rho}_{\text{out}} = \int \mathrm{d}t' \, P(t,t') \, \rho(t') \,, \quad (6)$$

where $P(t,t')$ will be specified later on.

Now, in order to see to what extend the real physical process approaches the ideal one, we use as parameter the fidelity

$$\mathcal{F} = \mathrm{Tr}\left\{\rho_{\text{out}} \, \overline{\rho}_{\text{out}}\right\}_{\text{ave}} , \quad (7)$$

where the subscript "ave" indicates the average overall possible input states. By making isotropic assumptions on these input states, it is possible to get

$$\mathcal{F} = \frac{3}{2^N[2^N+2]} \sum_{i=1}^{2^N} F_{i\,i}^{i\,i} + \frac{1}{2^N[2^N+2]} \sum_{i \neq j=1}^{2^N} \left( F_{i\,i}^{j\,j} + F_{i\,j}^{j\,i} \right) , \quad (8)$$

where

$$F_{j'j}^{i'i} = \int \mathrm{d}t' P(t,t') \langle j'|U(t'-t)|i\rangle\langle i'|U^\dagger(t'-t)|j\rangle . \quad (9)$$

In the case of a QPG we have

$$U(t)\,|j\rangle = \exp\left[-i\,\delta_{j,2^N}\,t\right]\,|j\rangle \,, \quad (10)$$

where $\delta_{j,j'}$ is the Kronecker symbol. A straightforward calculation gives

$$\mathcal{F} = \frac{[2^{2N}+2] + 2[2^N-1]\mathrm{Re}\{\mathcal{I}\}}{2^N[2^N+2]} \quad (11)$$

with

$$\mathcal{I} = \int \mathrm{d}t' \, P(t,t') e^{i(t'-t)} . \quad (12)$$

It is worth noting that $\mathrm{Re}\{\mathcal{I}\} \in \{-1, 1\}$ provided the distribution $P$ is normalized.

Typically, to account for dephasing errors it is quite natural to choose a Gaussian distribution [6], i. e.

$$P(t,t') = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(t'-t)^2}{2\sigma^2}\right] , \quad (13)$$

where $\sigma^2$ denotes the variance, while the average is $\langle t' \rangle = t$. Equation (13) immediately gives $\mathcal{I} = e^{-\sigma^2/2}$.

However, a more refined formalism describing non-dissipative decoherence has recently been developed [11]. It is based on the idea that time is a random variable or, alternatively, that the system Hamiltonian (therefore its eigenvalues) fluctuates. This leads to random phases in the energy eigenstates representation. Then, the resulting evolution of the system must be averaged on a suitable probability distribution, and this leads to the decay of the off-diagonal elements of the density operator. In [11], the function $P(t,t')$ has been determined to satisfy the following conditions: $\overline{\rho}(t)$ must be a density operator (it must be self-adjoint, positive-definite, and with unit-trace), and its time evolution must satisfy the semigroup property. All that has led to

$$P(t,t') = \frac{1}{\tau}\frac{e^{-t'/\tau}}{\Gamma(t/\tau)}\left(\frac{t'}{\tau}\right)^{(t/\tau)-1}, \tag{14}$$

where the parameter $\tau$ naturally appears as a scaling time. The expression (14) is the so-called $\Gamma$-distribution function. Generally, the meaning of the parameter $\tau$ can be understood by considering the average of the evolution time $\langle t'\rangle = t$ and its variance $\langle t'^2\rangle - \langle t'\rangle^2 = \tau t$. Hence, $\tau$ represents the strength of phase fluctuations. With respect to the Gaussian distribution we have the same avarage, but a time dependent variance, or better to say, a variance related to the mean value. In the limit $t \gg \tau$, (14) tends to a Gaussian shape, but still has a variance depending on the avagage.

By using the distribution (14) in (12), we get $\mathcal{I} = (1 - i\tau)^{-t/\tau}\, e^{-it}$. Note that in this case the integration (12) is performed in $\{0, \infty\}$ (as time should run) differently from the Gaussian case where the whole real interval is used.

Now, suppose to realize the transformation (2) through $M$ steps, i.e. $\pi = M\,t$, with $t$ a fixed time interval. Then, the fidelity (11) becomes, by composition,

$$\mathcal{F} = \frac{2^{2N}+2}{2^N[2^N+2]} + 2\frac{2^N-1}{2^N[2^N+2]}e^{-M\sigma^2/2} \tag{15}$$

for the Gaussian distribution (13), and

$$\mathcal{F} = \frac{2^{2N}+2}{2^N[2^N+2]} + 2\frac{2^N-1}{2^N[2^N+2]} \cdot \exp\left[-\frac{\pi}{\tau}\log\left(1+\tau^2\right)\right]\cos\left[\pi\left(\frac{\arctan\tau}{\tau}-1\right)\right] \tag{16}$$

for the Gamma distribution (14).

Essentially (15), (16) have the same dependence on the dimension of the Hilbert space, in the sense that the fidelity tends to 1 when $N \to \infty$ irrespectively of the decoherence. This is due to the possibility of having entanglement in high dimensional space. Nevertheless, (15) and (16) differ. First of all, in the Gaussian case the fidelity goes down by increasing the number of operations, $M$, while in the other case it only depends on the total time elapsed $\pi$. Furthermore, in the limit of large fluctuations, i.e. $\sigma \to \infty$ and $\tau \to \infty$, (15) reduces to $[2^{2N}+2]/2^N[2^N+2]$, while (16) yields $\{[2^{2N}+2]-2[2^N-1]\}/2^N[2^N+2]$ which is a worse limit. It happens that in the first case $\mathrm{Re}\{\mathcal{I}\}$ averages to 0, while in the second it averages to $-1$. This different behavior must be ascribed to the quantum mechanical consistency of the framework involving the Gamma distribution. Instead, the approach based on the Gaussian distribution is rather phenomenological. Moreover, the limit case of $\tau \to \infty$ is quite interesting, since it means that the phase fluctuations completely inhibits the gate operations [12].

Finally, the generality of the approach developed in [11] suggest the possibility that the parameter $\tau$, even though system-dependent, might have a lower nonzero limit which would be reached just in case of no fluctuations of experimental origin. Its *intrinsic* nature could be taken back to the energy-time uncertainty relation. However, the use of one or theother approache depends also on the physical system one is going to consider.

Summarizing, we have seen how dephasing errors affects a QPG reducing its versatility. To this end we have used, and compared, two different approaches. Probably, one should also design different codes to correct such errors [13] depending on the used framework.

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